A Brief on Wave Propagation in Solids

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What is This Note About?

- Terminology: phasor, wave front, wave number, phase velocity, and group speed
- Wave propagation in:
  - Strings
  - Bars
  - Beams
  - Shafts
  - Plates
- Phase and group velocity
- Non-dispersive and dispersive waves
- Coincidence frequency

Phasor

- Phasor is a rotating vector.
  - *Consider a vector **A** such that, **A** = a + bj is* a vector with *a*, and *b* as its real and imaginary components.
  - In exponential form, this vector may be expressed as, where:
    - $|\mathbf{A}| = (a^2 + b^2)^{0.5}$
    - $= \mathrm{Tan}^{-1}(b/a)$
  - If this vector is rotating at an angular velocity of ω, then such a vector is called phasor.
  - Mathematically, this implies multiplication of vector ***A*** with a time-dependent vector $e^{j\omega t}$. The product of these two entities becomes a phasor.
  - Thus, $\mathbf{B} = |A| e^{j\phi} \text{ X } e^{j\omega t} = |A| e^{j(\phi + \omega t)}$.

- Such rotating vectors are very frequently used in wave-mechanics.

- The projection of a phasor on real axis represents a real harmonic function.

Wave-front, and wavenumber

- Wave-front is a locus of points in a wave, that have the same phase.
  - For 3-D planar waves, these fronts are flat planes, each plane being parallel to the all other planes.
  - For 2-D planar waves, these fronts are straight lines, all being mutually parallel.
  - For radialy symmetric waves, as in those emitted by monopoles in 3-D space, these fronts may be represented by a series of concentric spheres.

- Lenses and mirrors may be used to transform planar wave fronts to spherical wave-fronts.

- Similarly, lenses and mirrors may also be used to transform spherical wave fronts to planar wave-fronts.

Wavenumber, Phase Velocity & Group Speed

- Wavenumber k, is defined as phase change per unit length. Since phase changes by 2π radians over one wavelength, thus, the per-unit length phase change k is defined as:
  - $k = 2\pi/\lambda$

- In a homogenous 1-D medium, if a point 'O' is disturbed simple harmonically with angular frequency ω, then this disturbance propagates away from the source at a speed c_{ph}. If the phase of this disturbance at O is φ, then the propagation velocity of phase away from O is also at the same speed, i.e. c_{ph}. *This velocity, c_{ph}, is termed as phase velocity.* The relation between k, and c_{ph} is:
- $k = 2\pi/\lambda = \omega/c_{ph}$.

- Group speed c_g of a wave is defined as $\partial\omega/\partial k$. Knowledge of c_g is useful in understanding wave energy flow.

1-D Waves Propagation in Solids

- Just as in fluid media, waves can propagate in solid medium as well. This propagation can be 1-D, 2-D or 3-dimensional in nature.

- Examples of one-dimensional wave propagation in solids are:
  - Vibration of a string
  - Longitudinal wave in 3-D solids
  - Quasi-longitudinal waves in thin plates
  - Transverse shear waves in solids
  - Longitudinal waves in a bar
  - Torsional waves in a bar
  - Bending waves in a bar

1-D Waves Propagation in Solids

- The governing equations for all these examples of 1-D waves in solids are given in following table.

Case	Governing Equation	Definition of Ψ	Phase velocity
String	$\partial^2\Psi/\partial x^2 = (1/c^2)\partial^2\Psi/\partial t^2$	String's normal displacement	$c^2 = T/\rho_L$ T = *tension in string* ρ_L = string's linear density
Longitudinal waves in a 3-D solid	$\partial^2\Psi/\partial x^2 = (1/c^2)\partial^2\Psi/\partial t^2$	Displacement in *x*-direction	$c^2 = E(1-\nu)/[\rho(1+\nu)(1-2\nu)]$ E = Young's modulus ν = Poisson's ratio ρ = material density
Quasi-longitudinal wave in thin plate	$\partial^2\Psi/\partial x^2 = (1/c^2)\partial^2\Psi/\partial t^2$	Displacement in *x*-direction	$c^2 = E/[\rho(1-\nu^2)]$ E = Young's modulus ρ = material density
Transverse shear waves solids	$\partial^2\Psi/\partial x^2 = (1/c^2)\partial^2\Psi/\partial t^2$	Transverse displacement in *y*-direction	$c^2 = G/\rho$ G = Shear modulus ρ = material density
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1-D Waves Propagation in Solids

Case	Governing Equation	Definition of Ψ	Phase velocity
Longitudinal waves in bar	$\partial^2\Psi/\partial x^2 = (1/c^2)\partial^2\Psi/\partial t^2$	Displacement in *x*-direction	$c^2 = E/\rho$ E = Young's modulus ρ = material density
Torsion waves in a bar	$\partial^2\Psi/\partial x^2 = (1/c^2)\partial^2\Psi/\partial t^2$	Twist angle	$c^2 = G/\rho$ G = Shear modulus ρ = material density
Bending waves in a bar	$\partial^4\Psi/\partial x^4 = [\rho_L/(EI)]\partial^2\Psi/\partial t^2$	Transverse displacement in *y*-direction	$c^2 = [EI\omega^2/\rho_L]^{(1/4)}$ E = Young's modulus I = Bending MOI ρ_L = bar's linear density ω = Angular frequency

- It is seen that the governing equation for all the cases, except that for bending waves in a bar, are similar.

- For all these cases, phase velocity is independent of angular frequency, ω.

Dispersive and Non-Dispersive Waves

- However, for the case of bending waves in a bar, phase velocity is directly proportional to the square root of angular frequency.

- This dependence of phase velocity on angular frequency is attributable to the fact that the governing equation is a 4^{th} order PDE in *x*, while those for all the remaining cases are 2^{nd} order PDEs in x.

- Waves with varying phase velocities with respect to angular frequencies are called *dispersive waves*. A "group" of these waves with different frequencies, may start travelling at the same time in a medium, but due to their different phase velocities, "disperse" as they travel along the medium.

- Waves with same phase velocities with respect to angular frequencies are called *non-dispersive waves*.

Coincidence

- The plot of wave number (on vertical axis) vis-à-vis frequency is called a dispersion curve.

- For non-dispersive waves, this curve is a straight line. For such waves, phase speed and group speed are identical. For dispersive waves, this curve is not a straight line.

- Bending waves in plates are dispersive waves. Hence, at certain frequency, the phase speed of these waves "coincides" with phase speed of sound in fluid (air). This frequency is called *coincidence frequency*.

- On a dispersion plot, *coincidence frequency* corresponds to the point of intersection of dispersion curve for a bending wave (in a plate) and dispersion curve for sound (a straight line).